Local behaviour of first passage probabilities

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Abstract

Suppose that S is an asymptotically stable random walk with norming sequence c_n and that T_x is the time that S first enters (x,∞), where $x\geq 0$. The asymptotic behaviour of $P(T_0=n)$ has been described in a recent paper of Vatutin and Wachtel [21], and here we build on that result to give three estimates for $P(T_x=n)$, which hold uniformly as $n\to\infty$ in the regions $x=o(c_n)$, $x=O(c_n)$, and $x/c_n\to\infty$, respectively.

1 Introduction

Supppose S is a 1-dimensional random walk and for $x\geq 0$ let T_x be the first exit time of $(-\infty,x]$, and write T for T_0: thus T is also the first strict ascending ladder time in S. Results about the tail behaviour of T_x are known in three different regimes. Firstly, with U denoting the renewal function in the strict increasing ladder process of S, and with x denoting any **fixed** continuity point of U**,** for any $\rho\in(0,1)$ the following statements are equivalent:

$$P(S_n>0)\to\rho \text{ as } n\to\infty: \tag{1}$$

$$P(T_x>n)\backsim U(x)n^{-\rho}L(n) \text{ as } n\to\infty. \tag{2}$$

(Here L denotes a function which is slowly varying (s.v.) at ∞; its asymptotic behaviour is determined by the sequence $(\rho_n, n\geq 1)$, where $\rho_n=P(S_n>0)$, see e.g. [10].) The case $x=0$ of (2) asserts that T is in the domain of attraction of a positive stable law of index $\rho:$ we write this as $T\in D(\rho,1)$.

In particular, (1) and (2) hold in the situation that S is in the domain of attraction of a strictly stable law without centreing (we write $S\in D(\alpha,\rho)$, where $\alpha\in(0,2]$ is the index and $\rho\in(0,1)$ is the positivity parameter). In this asymptotically stable case, if c_n is such that $(S_{[nt]}/c_n,t\geq 0)\xrightarrow{d}(Y_t,t\geq 0)$, we can deduce from the functional central limit theorem that, when $x_n:=x/c_n$ is bounded away from zero and infinity,

$$(T_x>n)\backsim P(\sigma_{x_n}>1)=\int_1^\infty h_{x_n}(t)dt, \tag{3}$$

where $h_a(\cdot)$ is the density function of σ_a, the first passage time of the limiting stable process Y over a. Finally, if $\alpha\rho<1$, so that $\overline{F}$, the right-hand tail of the

distribution function of S_1, is regularly varying with index $-\alpha$, with $\alpha \in (0,2)$, (we write this as $\overline{F} \in RV(-\alpha)$), and $x/c_n \to \infty$, then it is known that

$$P(T_x \le n) = P(\max_{r \le n} S_r > x) \backsim n\overline{F}(x). \tag{4}$$

In this paper we will prove that in this asymptotically stable case **local** uniform versions of (2), (3), and (4) hold in the respective scenarios

$$\begin{array}{ll} A: & x/c_n \to 0, \\ B: & x/c_n \text{ is bounded away from 0 and } \infty, \\ C: & x/c_n \to \infty. \end{array}$$

The inspiration for this programme comes from a recent paper by Vatutin and Wachtel [21], who show that in almost all cases that $S \in D(\alpha, \rho)$ the following local estimate holds:

$$P(T = n) \backsim \rho n^{-\rho-1} L(n) \text{ as } n \to \infty \ . \tag{5}$$

(They actually show that (5) can only fail if S lives on a non-centred lattice, when a modified version holds: we do not treat this case.) The statement (5) is a local version of the special case $x = 0$ of (2), and we mention at this point that their proof is quite different according as $\alpha\rho < 1$ or $\alpha\rho = 1$. We also mention that prior to [21], the asymptotic behaviour of $P(T = n)$ was apparently only known in the case of attraction to the Normal distribution: see [17] and [3]. However the asymptotic behaviour of the ratio $P(T_x = n)/P(T = n)$ for fixed x is known for strongly aperiodic recurrent random walk on the integers, (see Theorem 7 of [20]), so our focus is mainly on the case that $x \to \infty$.

Our first result shows that the obvious local version of (2), viz

$$P(T_x = n) \backsim \rho U(x) n^{-\rho-1} L(n) \text{ as } n \to \infty, \tag{6}$$

holds uniformly for $x \ge 0$ in case A.

In case B, our result is a uniform local version of (3), which is valid in all cases.

Finally in case C, we prove a uniform local version of (4), but this requires the additional assumption that $\alpha\rho < 1$, so that $\overline{F} \in RV(-\alpha)$, and also a local version of this assumption.

To the best of our knowledge, these results are new for non-constant x, except for the case of finite variance, where similar results were established in Eppel [17].

Our method of proof in cases A and B relies crucially on several different local estimates of the distribution of S_n conditional on $T_x > n$, which extend results for the case $x = 0$ from [21], and in case C we use a conditional local limit theorem from [19].

We state our notation, assumptions and results in detail in the next section, then give some preliminary results in section 3, prove the above-mentioned estimates, which may be of independent interest, in section 4, give a full proof of our main results in the lattice case in section 5, and sketch the proof in the non-lattice case in the final section.

2 Results

Notation In what follows the phrase "S is an a.s.r.w.", (asymptotically stable random walk) will have the following meaning.

- $S = (S_n, n \geq 0)$ is a 1-dimensional random walk with $S_0 = 0$ and $S_n = \sum_1^n X_r$ for $n \geq 1$ where $X_1, X_2, \cdots$ are i.i.d. with $F(x) = P(X_1 \leq x)$:
- S is either non-lattice, or it takes values on the integers and is aperiodic:
- there is a monotone increasing continuous function $c(t)$ such that the process defined by $X_t^{(n)} = S_{[nt]}/c_n$ converges weakly as $n \to \infty$ to a stable process $Y = (Y_t, t \geq 0)$.
- the process Y has index $\alpha \in (0, 2]$, and $\rho := P(Y_1 > 0) \in (0, 1)$.

Remark 1 *The case $\alpha\rho = 1, \alpha \in (1, 2]$ is the spectrally negative case, and we will sometimes need to treat this case separately. If $\alpha\rho < 1$ then $\alpha < 2$ and the Lévy measure Π of Y has a density equal to $c_+ x^{-\alpha-1}$ on $(0, \infty)$ with $c_+ > 0$, and then we can also assume that the norming sequence satisfies*

$$n\overline{F}(c_n) \to 1 \text{ as } n \to \infty. \tag{7}$$

But if $\alpha\rho = 1$ we will have

$$n\overline{F}(c_n) \to 0 \text{ as } n \to \infty. \tag{8}$$

Here are our main results, where we recall that $h_y(\cdot)$ denotes the density function of the passage time over level $y > 0$ of the process Y. We will also adopt the convention that both x and Δ are restricted to the integers in the lattice case.

Theorem 2 *Assume that S is an asrw. Then*
(A) uniformly for x such that $x/c_n \to 0$,

$$P(T_x = n) \backsim U(x)P(T = n) \backsim \rho U(x) n^{-\rho-1} L(n) \text{ as } n \to \infty : \tag{9}$$

(B) uniformly in $x_n := x/c(n) \in [D^{-1}, D]$, for any $D > 1$,

$$P(T_x = n) \backsim n^{-1} h_{x_n}(1) \text{ as } n \to \infty. \tag{10}$$

If, in addition, $\alpha\rho < 1$, and

$$f_x^{\Delta} := P(S_1 \in [x, x + \Delta)) \text{ is regularly varying as } x \to \infty, \tag{11}$$

then
(C) uniformly for x such that $x/c_n \to \infty$,

$$P(T_x = n) \backsim \overline{F}(x) \text{ as } n \to \infty. \tag{12}$$

From this we get immediately a strengthening of (2).

Corollary 3 *If S is an a.s.r.w. the estimate*

$$P(T_x > n) \backsim U(x)n^{-\rho}L(n) \text{ as } n \to \infty$$

holds uniformly as $x/c_n \to 0$.

Remark 4 *In view of (3) the result (10) might seem obvious. However (3) could also be written as $P(\max_{r \leq n} S_r \leq x) \backsim \int_0^{x_n} m(y)dy$, where m denotes the density function of $\sup_{t \leq 1} Y_s$, and in the recent paper [22], Wachtel has shown that the obvious local version of this is only valid under an additional hypothesis.*

Remark 5 *The asymptotic behaviour of $h_x(1)$ has been determined in [16], and is given by*

$$h_x(1) \backsim k_1 x^{\alpha\rho} \text{ as } x \downarrow 0, \; h_x(1) \backsim k_2 x^{-\alpha} \text{ as } x \to \infty.$$

(We mention here that $k_1, k_2, \cdots$ will denote particular fixed positive constants whereas C will denote a generic positive constant whose value can change from line to line.) It is therefore possible to compare the exact results in (9) and (12) with the behaviour of $n^{-1}h_{x_n}(1)$ when $x/c_n \to 0$ or $x/c_n \to \infty$. It turns out that the ratio of the two can tend to 0, or a finite constant, or oscillate, depending on the s.v. functions involved and the exact behaviour of x, except in the special case that $c_n \backsim Cn^{\eta}$. (Here, and throughout, we write $1/\alpha = \eta$.) In fact, if $\overline{F}(x) \backsim C/(x^{\alpha}L_0(x))$, one can check that, when $x/c_n \to \infty$,

$$\frac{n\overline{F}(x)}{h_{x_n}(1)} \backsim \frac{L_0(c_n)}{L_0(x)},$$

and of course L_0 is asymptotically constant only in the aforementioned special case. Similarly, the RHS of (9) only has the same asymptotic behaviour as $n^{-1}h_{x_n}(1)$ in this same special case.

Remark 6 *In the spectrally negative case $\alpha\rho = 1$, without further assumptions we know little about the asymptotic behaviour of $\overline{F}$, so it is clear that (C) doesn't generally hold in this case, and indeed it is somewhat surprising that parts (A) and (B) do hold.*

3 Preliminaries

Throughout this section it will be assumed that S is an a.s.r.w.. With $(\tau_0, H_0) := (0,0)$ we write $(\boldsymbol{\tau}, \boldsymbol{H}) = ((\tau_n, H_n), n \geq 0)$ for the bivariate renewal process of strict ladder times and heights, so that $\tau_1 = T$ and $H_1 = S_T$ is the first ladder height; we also write τ and H for τ_1 and H_1. It is known that there are sequences a_n and b_n such that $(\tau_n/a_n, H_n/b_n)$ converges in distribution to a bivariate law whose marginals are positive stable laws with parameters ρ and

$\alpha\rho$ respectively, with the proviso that when $\alpha\rho = 1$ we replace the stable limit of H_n/b_n by a point mass at 1. Thus a, b, c are regularly varying with indexes $\rho^{-1}, (\alpha\rho)^{-1}$, and η respectively. Furthermore we can assume, without loss of generality, the existence of continuous, increasing functions a, b, c such that $a_n = a(n), b_n = b(n), c_n = c(n)$, and

$$b(t) = k_3 c(a(t)),\ t \geq 0. \tag{13}$$

(See [14] for details).

Write $A(y) = \int_0^\infty P(H > y)dy$. We will find the following consequence of (13) useful.

Lemma 7 *There is a constant k_4 such that*

$$U(c_n) \backsim \frac{c_n}{A(c_n)} \backsim \frac{k_4}{P(\tau > n)} \text{ as } n \to \infty. \tag{14}$$

Proof. The first statement is due to Erickson [18], and the second is a slight reformulation of Lemma 13 of [21], using the fact that $nP(\tau > n)P(\tau^- > n) \to k_5$, where $\tau^- = \min\{n \geq 1 : S_n \leq 0\}$ is the first weak decreasing ladder time. ■

Corollary 8 *If V is the renewal function in the weak decreasing ladder height process then there is a constant k_5 such that*

$$U(c_n)V(c_n) \backsim k_6 n \text{ as } n \to \infty. \tag{15}$$

Proof. This follows from $nP(\tau > n)P(\tau^- > n) \to k_5$, (14), and the analogous statement about V. ■

We will also need the following conditional functional limit theorem, in which $X^{(n)}(t) = S_{\lfloor nt \rfloor}/c_n, 0 \leq t \leq 1$, where $\lfloor\cdot\rfloor$ denotes the integer part function, and S is an a.s.r.w.

Proposition 9 *Let P_x denote the probability measure under which S starts at $x \geq 0$, and put $T^- = \min(n \geq 1 : S_n \leq 0)$,*

(i) If $x/c_n \to 0$, then $P_x(X^{(n)} \in \cdot | T^- > t)$ converges weakly on the Skorohod space to $P(Z^{(1)} \in \cdot)$, where $Z^{(1)}$ denotes the stable meander of length 1 based on Y,

(ii) If $x/c_n \to a > 0$, then $P_x(X^{(n)} \in \cdot | T^- > t)$ converges weakly on the Skorohod space to $P(a + Y \in \cdot | a + \inf_{s \leq 1} Y_s > 0)$.

Proof. (i) This is proved in [11] for the special case $x \equiv 0$, and it is not difficult to deduce the quoted result by using the technique in Section 5 of [7]. (Actually the proof in [7] is for the case $\alpha = 2$, and concerns convergence to the Bessel process, rather than the Brownian meander: but these processes are mutually absolutely continuous, and so are their analogues for stable processes. We can therefore deduce convergence to the meander from convergence to the process conditioned to stay positive, as is done in a more general scenario in Section 4 of [8].)

(ii) Since the probability of the limiting conditioning event is positive, this follows from the weak convergence of $X^{(n)}$ to Y. ■

Until further notice we assume we are in the lattice case, and $x, y, z \cdots$ will be assumed to take non-negative integer values only.

We will write

$$g(m,y)=\sum_{n=0}^{\infty}P(T_n=m,H_n=y) \text{ and } g(y)=\sum_{n=0}^{\infty}P(H_n=y)$$

for the bivariate renewal mass function of $(\boldsymbol{\tau},\boldsymbol{H})$ and the renewal mass function of $\boldsymbol{H}$ respectively. Our proofs are based on the following obvious representation:

$$P(T_x=n+1)=\sum_{y\geq 0}P(T_x>n,S_n=x-y)\overline{F}(y),\ \ x\geq 0, n>0, \tag{16}$$

To exploit this we need good estimates of $P(T_x>n, S_n>x-y)$, and we derive these from the formula

$$P(S_n=x-y,T_x>n)=\sum_{z=0}^{y\wedge x}\sum_{r=0}^{n}g(r,x-z)g^-(n-r,y-z), \tag{17}$$

where g^- denotes the bivariate mass function in the weak downgoing ladder process of S. Formula (17), which extends a result originally due to Spitzer, follows by decomposing the event on the LHS according to the time and position of the maximum, and using the well-known duality result:

$$g^-(m,u)=P(S_m=-u,\tau>m). \tag{18}$$

(See Lemma 2.1 in [4]). Of course we also have

$$g(m,u)=P(S_m=u,\tau^->m), \tag{19}$$

and our main tool in estimating $P(S_n=x-y,T_x>n)$ will be the following estimates for g and g^-. The results for g are established in [21], where they are stated as estimates for the conditional probability $P(S_m=u|\tau^->m)$. (See Theorems 5 and 6 in [21].) The results for g^- can be derived by applying those results to $-S$, and then using the calculation given on page 100 of [3] to deduce the result for the weak ladder process. Recall that $V(x)=\sum_{m=0}^{\infty}\sum_{u=0}^{x}g^-(m,u)$, and write f for the density of Y_1, where Y is the limiting stable process. Also p and $\tilde{p}$ stand for the densities of $Z_1^{(1)}$ and $\tilde{Z}_1^{(1)}$, the stable meanders of length 1 at time 1 corresponding to Y and $-Y$.

Lemma 10 *Uniformly in $x\geq 1$ and $x\geq 0$, respectively,*

$$\frac{c_n g(n,x)}{P(\tau^->n)}=p(x/c_n)+o(1) \text{ and } \frac{c_n g^-(n,x)}{P(\tau>n)}=\tilde{p}(x/c_n)+o(1) \text{ as } n\to\infty. \tag{20}$$

Also, uniformly as $x/c_n\to 0$,

$$g(n,x)\backsim\frac{f(0)U(x-1)}{nc_n} \text{ for } x\geq 1 \text{ and } g^-(n,x)\backsim\frac{f(0)V(x)}{nc_n} \text{ for } x\geq 0. \tag{21}$$

From this we can deduce the following result, which is a minor extension of Lemma 20 in [21].

Lemma 11 *Given any constant C_1 there exists a constant C_2 such that for all $n \geq 1$ and $0 \leq x \leq C_1 c_n$*

$$g(n,x) \leq \frac{C_2 U(x)}{nc_n}, \text{ and } g^-(n,x) \leq \frac{C_2 V(x)}{nc_n}. \tag{22}$$

.

Proof. Observe that on any interval $[\delta c_n, C_1 c_n]$, $p(x/c_n)$ is bounded, and $U(x) \geq U(\delta c_n)$, so by Lemma 7 we see that the ratio

$$\frac{P(\tau^- > n)}{c_n} / \frac{U(x)}{nc_n} \leq nP(\tau^- > n)U(\delta c_n)$$

is also bounded above. A similar proof works for g^-. ■

Just as these local estimates for the distribution of S_n on the event $\tau > n$ played a crucial rôle in the proof of (5) in [21], we need similar information on the event $T_x > n$. This is given in the following result, where for $x > 0$ we write $q_x(\cdot)$ for the density of $P(Y_1 \in x - \cdot : \sup_{t \leq 1} Y_t < x)$. We also write $x/c_n = x_n$ and $y/c_n = y_n$.

Proposition 12 *(i) Uniformly as $x_n \vee y_n \to 0$*

$$P(S_n = x-y, T_x > n) \backsim \frac{U(x)f(0)V(y)}{nc_n}. \tag{23}$$

(ii) For any $D > 1$, uniformly for $y_n \in [D^{-1}, D]$,

$$P(S_n = x-y, T_x > n) \backsim \frac{U(x)P(\tau > n)\tilde{p}(y_n)}{c_n} \text{ as } n \to \infty \text{ and } x_n \to 0, \tag{24}$$

and uniformly for $x_n \in [D^{-1}, D]$,

$$P(S_n = x-y, T_x > n) \backsim \frac{V(y)P(\tau^- > n)p(x_n)}{c_n} \text{ as } n \to \infty \text{ and } y_n \to 0. \tag{25}$$

(iii) For any $D > 1$, uniformly for $x_n \in [D^{-1}, D]$ and $y_n \in [D^{-1}, D]$,

$$P(S_n = x-y, T_x > n) \backsim \frac{q_{x_n}(y_n)}{c_n} \text{ as } n \to \infty. \tag{26}$$

The proof of this result is given in the next section. We can repeat the argument used in Lemma 11 to get the following corollary.

Corollary 13 *Given any constant C_1 there exists a constant C_2 such that for all $n \geq 1$ and $0 \leq x \leq C_1 c_n, 0 \leq y \leq C_1 c_n$,*

$$P(S_n = x-y, T_x > n) \leq \frac{C_2 U(x) f(0) V(y)}{nc_n}.$$

It is apparent that we will also need information about the behaviour of $g(n,x)$, or equivalently of $P(S_n = x, \tau^- > n)$, in the case $x/c_n \to \infty$. Fortunately this has been obtained recently in [19], and we quote Propositions 11 and 12 therein as (28) and (30). The related unconditional results (27) and (29) have been proved in special cases in [12], [13], and [19], and the general results can be deduced from Theorem 2.1 of [9].

Proposition 14 *If S is an asrw with $\alpha\rho < 1$, then, uniformly for x such that $x/c_n \to \infty$,*

$$P(S_n > x) \backsim nP(S_1 > x) \text{ as } n \to \infty, \text{ and} \tag{27}$$

$$P(S_n > x, \tau^- > n) \backsim \rho^{-1}P(S_n > x)P(\tau^- > n) \text{ as } n \to \infty. \tag{28}$$

If, additionally, (11) holds, then

$$P(S_n \in [x, x+\Delta)) \backsim nf_x^{\Delta} \text{ as } n \to \infty \tag{29}$$

and

$$P(S_n \in [x, x+\Delta), \tau^- > n) \backsim \rho^{-1}nf_x^{\Delta}P(\tau^- > n) \text{ as } n \to \infty. \tag{30}$$

3.1 Some identities for stable processes.

Proposition 15 *(i) For any stable process Y which has $\alpha\rho < 1$ there are positive constants k_7 and k_8 such that the following identities hold:*

$$h_x(t) = k_7 \int_0^\infty q_x(w)w^{-\alpha}dw \text{ and} \tag{31}$$

$$q_u(v) = k_8 \int_0^1 \int_0^{u\wedge v} u(t, u-z)u^-(1-t, v-z)dzdt, \tag{32}$$

where u and u^- denote the bivariate renewal densities for the increasing ladder processes of Y and $-Y$.

(ii) If $\alpha\rho = 1$ there is a positive constant k_9 such that

$$p(x) = k_9 h_x(1). \tag{33}$$

Proof. All three results are special cases of results for Lévy processes. The general version of (31) is given in [15], and (32) follows from the following observation, which is a minor extension of Theorem 20, p176 of [5].

Assume X is a Lévy process which is not compound Poisson. Then there is a constant $k_7 > 0$ such that for $x > 0$ and $w < x$,

$$P(X_t \in dw, t < T_x)dt = k_7 \int_{y=w^+}^{x} \int_{s=0}^{t} U(ds, dy)U^-(dt-s, y-dw), \tag{34}$$

where U and U^- are the bivariate renewal measures in the increasing ladder processes of X and X^-. Clearly it suffices to prove that the Laplace transform,

in t, of the LHS of (34) is the same as that of the RHS, which is

$$\begin{aligned} & k_7 \int_{y=w^+}^{x} \int_{t=0}^{\infty} e^{-qt} \int_{s=0}^{t} U(ds,dy)U^-(dt-s,y-dw) \\ = \quad & k_7 \int_{y=w^+}^{x} \int_{t=0}^{\infty} e^{-qt} U(dt,dy) \int_{s=0}^{\infty} e^{-qs} U^-(ds,y-dw). \end{aligned} \tag{35}$$

Note that if we introduce an independent $\mathrm{Exp}(q)$ random variable $\boldsymbol{e}_q$ the Wiener-Hopf factorisation allows us to write $X_{\boldsymbol{e}_q} = S_{\boldsymbol{e}_q} - \widetilde{S}^-_{\boldsymbol{e}_q}$, where S denotes the supremum process of X and $\tilde{S}^-$ denotes an independent copy of the supremum process S^- of $-X$. Let κ and κ^- denote the bivariate Laplace exponents of the ladder processes of X and X^-.Then, using the identity $\kappa(q,0)\kappa^-(q,0) = q/k_7$ which follows from the Wiener-Hopf factorisation, (see e.g. (3), p166 of [5]),

$$\begin{aligned} \int_{t=0}^{\infty} e^{-qt} P(X_t \quad & \in \quad dw, t < T_x)dt \\ &= q^{-1} P(X_{\boldsymbol{e}_q} \in dw, \boldsymbol{e}_q < T_x) \\ &= q^{-1} P(S_{\boldsymbol{e}_q} \le x, S_{\boldsymbol{e}_q} - \widetilde{S}^-_{\boldsymbol{e}_q} \in dw) \\ &= q^{-1} \int_{w^+}^{x} P(S_{\boldsymbol{e}_q} \in dy) P(S^-_{\boldsymbol{e}_q} \in y - dw) \\ &= k_7 \int_{w^+}^{x} \frac{P(S_{\boldsymbol{e}_q} \in dy)}{\kappa(q,0)} \frac{P(S^-_{\boldsymbol{e}_q} \in y - dw)}{\kappa^-(q,0)}. \end{aligned}$$

But (1), p 163 of [5] gives

$$\frac{E(e^{-\lambda S_{\boldsymbol{e}_q}})}{\kappa(q,0)} = \frac{1}{\kappa(q,\lambda)} = \int_0^\infty \int_0^\infty e^{-(qt+\lambda y)} U(dt,dy),$$

so

$$\frac{P(S_{\boldsymbol{e}_q} \in dy)}{\kappa(q,0)} = \int_0^\infty e^{-qt} U(dt,dy).$$

Using the analogous expression for $P(S^-_{\boldsymbol{e}_q} \in y - dw)$, (35) is immediate, and then (34) follows. Specializing this to the stable case then gives (32). Finally, if we write n for the characteristic measure of the excursions away from zero of $X - I$, with I denoting the infimum process of X, then $p_t(dx) := n(\varepsilon_t \in dx)/n(\zeta > t)$ is a probabilty measure which coincides with that of the meander of length t at time t in the stable case. (Here ζ denotes the life length of the generic excursion ε.) In the special case of spectrally negative Lévy processes, we have

$$\begin{aligned} p_t(dx) &= Cxt^{-1} P(X_t \in dx)/n(\zeta > t) \\ &= Ch_x(t)dx/n(\zeta > t). \end{aligned}$$

The first equality here comes from Cor 4 in [2], and the second is the Lévy version of the ballot theorem (see Corollary 3, p 190 of [5]). Specialising to the stable case and $t = 1$ gives (33). (I owe this observation to Loic Chaumont.) ∎

4 Proof of Proposition 12

Proof. We will be applying the results in Lemma 10 to formula (17) and we write the RHS of (17) as $P_1 + P_2 + P_3$, where with $\delta \in (0, 1/2)$,

$$P_i = \sum_{r \in A_i} \sum_{z=0}^{y \wedge x} g(r, x-z) g^-(n-r, y-z), \ 1 \le i \le 3,$$

and

$$A_1 = \{0 \le r \le \delta n\}, \ A_2 = \{\delta n < r \le (1-\delta)n\}, \text{ and } A_3 = \{(1-\delta)n < r \le n\}.$$

(i) We introduce $d(\cdot)$, an increasing and continuous function which satisfies $d(n) = nc_n$, and with $\lfloor \cdot \rfloor$ standing for the integer part function, use Lemma 10 to write

$$\begin{aligned} P_1 &\sim f(0) \sum_{z=0}^{y\wedge x} \sum_{r=0}^{\lfloor n\delta \rfloor} g(r, x-z) \frac{V(y-z)}{d(n-r)} \\ &\le \frac{f(0)}{d(n(1-\delta))} \sum_{z=0}^{y \wedge x} V(y-z) \sum_{r=0}^{\lfloor n\delta \rfloor} g(r, x-z) \\ &= \frac{f(0)}{d(n(1-\delta))} \sum_{z=0}^{y\wedge x} V(y-z)[u(x-z) - \sum_{r=\lfloor n\delta \rfloor+1}^{\infty} g(r, x-z)]. \end{aligned}$$

We can apply Lemma 10 again to get the estimate, uniform for $w/c_n \to 0$,

$$\sum_{r=\lfloor n\delta \rfloor+1}^{\infty} g(r, w) \sim \sum_{r=\lfloor n\delta \rfloor+1}^{\infty} \frac{f(0)}{rc_r} U(w) \sim \frac{Cf(0)}{c_n} U(w).$$

Since we know that

$$\lim \inf_{n\to\infty} \frac{nu(n)}{U(n)} > 0,$$

(see e.g. Theorem 8.7.4 in [6]) we see that this is $o(u(w))$. Also we have

$$\sum_{z=0}^{y\wedge x} \sum_{r=0}^{\lfloor n\delta \rfloor} g(r, x-z) \frac{V(y-z)}{d(n-r)} \ge \frac{1}{d(n)} \sum_{z=0}^{y\wedge x} \sum_{r=0}^{\lfloor n\delta \rfloor} g(r, x-z) V(y-z),$$

so we see that

$$\lim_{n,\delta} \frac{d(n)P_1}{f(0)\sum_{z=0}^{y\wedge x} V(y-z)u(x-z)} = 1, \tag{36}$$

where $\lim_{n,\delta}(\cdot) = 1$ is shorthand for $\lim_{\delta\to 0}\limsup_{n\to\infty}(\cdot) = \lim_{\delta\to 0}\liminf_{n\to\infty}(\cdot) = 1$. Similarly, once we observe that

$$\begin{aligned} P_3 &= \sum_{z=0}^{y\wedge x}\sum_{r=n-\lfloor n\delta\rfloor}^{n} g(r, x-z)g^-(n-r, y-z) \\ &= \sum_{z=0}^{y\wedge x}\sum_{r=0}^{\lfloor n\delta\rfloor} g(n-r, x-z)g^-(r, y-z), \end{aligned}$$

an entirely analogous argument gives

$$\lim_{n,\delta}\frac{d(n)P_3}{f(0)\sum_{z=0}^{y\wedge x}U(x-z-1)v(y-z)} = 1, \tag{37}$$

where v is the renewal mass function in the down-going ladder height process. Noting that

$$U(x-z-1)v(y-z)+V(y-z)u(x-z) = U(x-z)V(y-z)-U(x-z-1)V(y-z-1),$$

we get the formula

$$\sum_{z=0}^{y\wedge x}\{U(x-z-1)v(y-z)+V(y-z)u(x-z)\} = V(y)U(x),$$

and the result will follow by letting $n\to\infty$ and then $\delta\downarrow 0$ provided $d(n)P_2 = o(V(y)U(x))$ for each fixed $\delta > 0$. In fact, using Lemma 10 again,

$$\begin{aligned} d(n)P_2 &= d(n)\sum_{A_2}\sum_{z=0}^{y\wedge x} g(r, x-z)g^-(n-r, y-z) \\ &\sim f(0)^2 d(n)\sum_{z=0}^{y\wedge x}\sum_{A_2}\frac{V(y-z)U(x-z)}{d(r)d(n-r)} \\ &\le \frac{(y\wedge x)f(0)^2 d(n)nV(y)U(x)}{d(\lfloor n\delta\rfloor)^2} = O\Big(\frac{(y\wedge x)V(y)U(x)}{c_n}\Big) = o(V(y)U(x)), \end{aligned}$$

and the result follows.

(ii) In this case we can assume WLOG that $y\wedge x = x = o(y)$, so that Lemma 10 gives $g^-(n-r, y-z) \backsim P(\tau > n-r)\tilde{p}(y/c_{n-r})/c_{n-r}$ uniformly for $r\in A_1$and $0\le z\le x$. With e denoting a continuous and monotone interpolant of $c_m/P(\tau > m)$, and noting that $\tilde{p}(\cdot)$ is uniformly continuous and bounded away from zero and infinity on $[D^{-1}, D]$, see [16], we can use a similar argument

to that in (i) to show that

$$
\begin{aligned}
P_1 &= \sum_{0}^{\lfloor \delta n \rfloor} \sum_{z=0}^{x} \frac{\tilde{p}(y/c_{n-r})}{e(n-r)} g(r, x-z)(1+o(1)) \\
&\le \frac{\tilde{p}^*(y)}{e(n(1-\delta))} \left(\sum_{z=0}^{x} u(x-z)(1+o(1)) \right), \\
&= \frac{U(x)\tilde{p}^*(y)}{e(n(1-\delta))}(1+o(1)),
\end{aligned}
$$

where $\tilde{p}^*(y) = \sup_{0\le r\le \delta n} \tilde{p}(y/c_{n-r}) = \tilde{p}(y_n) + \varepsilon(n,\delta)$ and $\lim_{n,\delta} \varepsilon(n,\delta) = 0$. In the same way we get $P_1 \ge U(x)\tilde{p}_*(y)/e(n)(1+o(1))$, where $\tilde{p}_*(y) = \inf_{0\le r\le \delta n} \tilde{p}(y/c_{n-r})$, and we deduce that

$$
\lim_{n,\delta} \frac{e(n)P_1}{\tilde{p}(y_n)U(x)} = 1. \tag{38}
$$

Since $inf\{\tilde{p}(y) : y \in [D^{-1}, D] > 0$, the result will follow if we can show that for any fixed $\delta > 0$

$$
\limsup_{n\to\infty} \frac{e(n)(P_2+P_3)}{U(x)} = 0. \tag{39}
$$

However (37) still holds, but note now that

$$
\begin{aligned}
\sum_{z=0}^{y\wedge x} U(x-1-z)v(y-z) &= \sum_{z=0}^{x} U(x-1-z)v(y-z) \\
&\le U(x)(V(y) - V(y-x-1)) = o(U(x)V(y)).
\end{aligned}
$$

and since the analogue of (14) holds, viz $V(c_n) \backsim k_{10}/P(\tau^- > n)$, we see that

$$
\begin{aligned}
\frac{e(n)P_3}{U(x)} &= o\left(\frac{U(x)V(y)\,e(n)}{d(n)U(x)} \right) \\
&= o(\frac{1}{nP(\tau > n)P(\tau^- > n)}) = o(1).
\end{aligned}
$$

Finally

$$
\begin{aligned}
e(n)P_2 &= e(n)\sum_{A_2}\sum_{z=0}^{x} g(r, x-z)g^-(n-r, y-z) \\
&\sim f(0)e(n)\sum_{A_2}\sum_{z=0}^{x} \frac{U(x-z)\tilde{p}((y-z)/c_{n-r})}{d(r)e(n-r)} \\
&\le \frac{Ce(n)nxU(x)}{d(\lfloor n\delta \rfloor)e(\lfloor n\delta \rfloor)} = O(\frac{xU(x)}{c_n}) = o(U(x)).
\end{aligned}
$$

Thus (39) is established, and the result (24) follows. Since (25) is (24) for $-S$ with x and y interchanged, modified to take account of the difference between strict and weak ladder epochs, we omit it's proof.

(iii) In this case it is P_2 that dominates. To see this, note that if we denote by $b(\cdot)$ a continuous interpolant of $c_n/P(\tau^- > n)$, we have $b(n)e(n) \backsim nc_n^2$. Then using Lemma 10 twice gives, uniformly for $x_n, y_n \in (D^{-1}, D)$ and for any fixed $\delta \in (0, 1/2)$,

$$\begin{aligned} c_n P_2 &= c_n \sum_{A_2} \sum_{z=0}^{x\wedge y} \frac{p((x-z)/c_r)\tilde{p}((y-z)/c_{n-r})}{b(r)e(n-r)} + o(c_n(x\wedge y)\sum_{A_2} \frac{1}{b(r)e(n-r)}) \\ &= c_n \sum_{A_2} \sum_{z=0}^{x\wedge y} \frac{p((x-z)/c_r)\tilde{p}((y-z)/c_{n-r})}{b(r)e(n-r)} + o(1). \end{aligned}$$

Making the change of variables $r = nt, y = c_n z$, recalling that p and $\tilde{p}$ are uniformly continuous on compacts, and that

$$\frac{b(r)e(n-r)}{b(n)e(n)} \to t^{-\eta-1+\rho}(1-t)^{-1-\rho}$$

uniformly on A_2, we see that for each fixed $\delta > 0$ we get the uniform estimate $c_n P_2 = I_\delta(x_n, y_n) + o(1)$, where

$$I_\delta(u,v) = \int_\delta^{1-\delta} \int_0^{u\wedge v} p(\frac{u-z}{t^\eta})\tilde{p}(\frac{v-z}{(1-t)^\eta})t^{-\eta-1+\rho}(1-t)^{-\eta-\rho}dtdz.$$

If we introduce $p_t(z) = t^{-\eta}p(zt^{-\eta})$, which is the density function of Z_t, the meander of length t at time t, according to Lemma 8 of [16] we have that the renewal measure of the increasing ladder process of Y has a joint density which is given by $u(t,z) = Ct^{\rho-1}p_t(z) = Ct^{-\eta+\rho-1}p(zt^{-\eta})$. Similarly for the decreasing ladder process we have $u^-(t,z) = C\, t^{-\eta+\rho}\tilde{p}(zt^{-\eta})$, and (32) in Proposition 15 gives

$$I_0(u,v) = C\int_0^1 \int_0^{u\wedge v} u(t, u-z)u^-(1-t, v-z)dzdt = Cq_u(v),$$

so we conclude that

$$\lim_{\delta\to 0}\lim_{n\to\infty} \frac{c_n P_2}{q_{x_n}(y_n)} = C. \tag{40}$$

Turning to P_1, if $K = \sup_{y\geq 0}\tilde{p}(y)$, we have

$$\begin{aligned} c_n P_1 &\backsim c_n \sum_0^{\lfloor \delta n\rfloor} \sum_{z=0}^{x\wedge y} \frac{\tilde{p}((y-z)/c_{n-r})}{e(n-r)} g(x-z,r) \\ &\leq \frac{Kc_n}{e(n(1-\delta))} \sum_0^{\lfloor \delta n\rfloor} \sum_{z=0}^{x} g(x-z,r) \leq CP(\tau > n)\Gamma(\lfloor \delta n\rfloor), \end{aligned}$$

where Γ is the renewal function in the increasing ladder time process. Since $T \in D(\rho, 1)$, we know that $\Gamma(\lfloor \delta n\rfloor) \backsim \delta^\rho \Gamma(n)$ and $P(\tau > n)\Gamma(n) \to C$, (see e.g.

p 361 of [6]), so we conclude that

$$\lim_{\delta\to 0}\lim\sup_{n\to\infty} c_n P_1 = 0.$$

Exactly the same argument applies to P_3, and since $q_u(v)$ is clearly bounded below by a positive constant for $u, v \in [D^{-1}, D]$ we have shown that (26) holds, except that the RHS is multiplied by some constant C. However if $C \neq 1$, by summing over y we easily get a contradiction, and this finishes the proof. ■

5 Proof of Theorem 2

5.1 Proof when $x/c_n \to 0$.

Proof. As already indicated, the proof involves applying the estimates in Proposition 12 to the representation (16), which we recall is

$$P(T_x = n+1) = \sum_{y\geq 0} P(S_n = x-y, T_x > n)\overline{F}(y),\ x \geq 0, n > 0. \tag{41}$$

In the case $\alpha\rho < 1$, given $\varepsilon > 0$ we can find K_ε and n_ε such that $n\overline{F}(K_\varepsilon c_n) \leq \varepsilon$ for $n \geq n_\varepsilon$ and, using (23) from Proposition 12,

$$P(S_n = x-y, T_x > n) \leq \frac{2U(x)f(0)V(y)}{nc_n} \text{ for all } x \vee y \leq \varepsilon c_n \text{ and } n \geq n_\varepsilon. \tag{42}$$

We can then use (24) of Proposition 12 to show that we can also assume, increasing the value of n_ε if necessary, that for $x \leq \varepsilon c_n$ and $y \in (\varepsilon c_n, K_\varepsilon c_n)$,

$$1-\varepsilon \leq \frac{c_n P(S_n = x-y, T_x > n)}{U(x)P(\tau > n)\tilde{p}(y_n)} \leq 1+\varepsilon \text{ for all } n \geq n_\varepsilon. \tag{43}$$

For fixed ε it is clear that as $n \to \infty$

$$\sum_{\varepsilon c_n < y < K_\varepsilon c_n} \tilde{p}(y_n)\overline{F}(y)/c_n \backsim \int_\varepsilon^{K_\varepsilon} \tilde{p}(z)\overline{F}(c_n z)dz \backsim n^{-1}\int_\varepsilon^{K_\varepsilon} \tilde{p}(z)z^{-\alpha}dz. \tag{44}$$

Since it is known (see (109) in [21] or Proposition 10 in [16]) that $k_{11} := \int_0^\infty z^{-\alpha}\tilde{p}(z)dz < \infty$, and we can assume $K_\varepsilon \uparrow \infty$ as $\varepsilon \downarrow 0$ we see that, provided

$$\lim_{\varepsilon\downarrow 0}\lim\sup_{n\to\infty} \frac{1}{U(x)P(\tau = n)} \sum_{y\in[0,\varepsilon c_n]\cup[K_\varepsilon c_n,\infty)} P(S_n > x-y, T_x > n)\overline{F}(y) = 0, \tag{45}$$

it will follow from (43) that $P(T_x = n) \backsim U(x)k_{11}\rho^{-1}P(\tau = n)$. Since this holds in particular for $x = 0$, we see that $k_{11} = \rho$, so it remains only to verify (45). Note first that

$$\begin{aligned}\sum_{y\in[K_\varepsilon c_n,\infty)} P(S_n &= x-y, T_x > n)\overline{F}(y) \leq \overline{F}(K_\varepsilon c_n)P(T_x > n)\\ &\leq \varepsilon n^{-1}P(T_x > n),\end{aligned}$$

so one part of (45) will follow if we can show that

$$P(T_x > n) \leq CU(x)P(\tau > n). \tag{46}$$

Obviously, for any $B > 0$

$$\begin{aligned} P(T_x &> n) = \sum_{y \geq 0} P(S_n = x - y, T_x > n) \\ &\leq \sum_{0 \leq y \leq Bc_n} P(S_n = x - y, T_x > n) + P(S_n < x - Bc_n, T_x > n). \end{aligned}$$

Since $x/c_n \to 0$, we can use the invariance principle in Proposition 9 to fix B large enough to ensure that $P(S_n < x - Bc_n | T_x > n) \leq 1/2$ for all sufficiently large n. We can also use Corollary 13 to get

$$\begin{aligned} \sum_{0 \leq y \leq Bc_n} P(S_n &= x - y, T_x > n) \leq \frac{CU(x)}{nc_n} \sum_{0 \leq y \leq Bc_n} V(y) \\ &\leq \frac{CU(x)Bc_n V(Bc_n)}{nc_n} \leq \frac{CU(x)}{nP(\tau^- > n)} \\ &\leq CU(x)P(\tau > n), \end{aligned}$$

and thus (46) is established. (Note that this proof is also valid for the case $\alpha\rho = 1$.) Now (42) gives

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} & \frac{1}{U(x)P(\tau = n)} \sum_{y \in [0, \varepsilon c_n]} P(S_n > x - y, T_x > n)\overline{F}(y) \\ &\leq \lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} \frac{2f(0)}{nc_n P(\tau = n)} \sum_{y \in [0, \varepsilon c_n]} \overline{F}(y)V(y), \end{aligned}$$

and since $\alpha\rho < 1$, (105) of [21] shows that this is zero.

In the case $\alpha\rho = 1$, a different proof is required. We make use of the observation in [21] that there is a sequence $\delta_n \downarrow 0$ with $\delta_n c_n \to \infty$ and such that $n\overline{F}(\delta_n c_n) \to 0$. This gives

$$\begin{aligned} \sum_{y > \delta_n c_n} P(S_n &= x - y, T_x > n)\overline{F}(y) \leq P(T_x > n)\overline{F}(\delta_n c_n) \\ &= o(U(x)P(\tau > n)/n) = o(U(x)P(\tau = n)), \end{aligned}$$

where we have used (46). Using (23) of Proposition 12 gives

$$\sum_{y \leq \delta_n c_n} P(S_n = x - y, T_x > n)\overline{F}(y) \backsim \frac{f(0)U(x)\omega(n)}{nc_n} \text{ as } n \to \infty,$$

uniformly in x, where $\omega(n) = \sum_{0 \leq y \leq \delta_n c_n} V(y)\overline{F}(y)$. But in [21], it is shown that $nc_n P(\tau = n) \backsim f(0)\omega(n)$, so we deduce from (41) that $P(T_x = n + 1) \backsim U(x)P(\tau = n)$, as required. ■

5.2 Proof when $x/c_n = O(1)$

Proof. Again we treat the case $\alpha\rho < 1$ first, and start by noting that for any $B > 0$,

$$\sum_{y>Bc_n} P(S_n = x - y, T_x > n)\overline{F}(y) \le \overline{F}(Bc_n) \backsim \frac{1}{nB^\alpha} \text{ as } n \to \infty,$$

uniformly in $x \ge 0$. Also by Corollary 13, for $b > 0$,

$$\sum_{y \le bc_n} P(S_n = x - y, T_x > n)\overline{F}(y) \le \frac{CU(x)\sum_{y\le bc_n} V(y)\overline{F}(y)}{nc_n}. \tag{47}$$

Since $V\overline{F} \in RV(-\alpha\rho)$, $\sum_{y\le z} V(y)\overline{F}(y) \backsim zV(z)\overline{F}(z)/(1-\alpha\rho)$, and we see that when $x \le Dc_n$,

$$\begin{aligned} \frac{U(x)\sum_{y\le bc_n} V(y)\overline{F}(y)}{c_n} &\backsim b^{1-\alpha\rho}U(x)V(c_n)\overline{F}(c_n) \\ &\le Cb^{1-\alpha\rho}D^{\alpha\rho}U(c_n)V(c_n)/n \le Cb^{1-\alpha\rho}D^{\alpha\rho}, \end{aligned}$$

where we have used (15) from Corollary 8. We conclude the proof by showing that

$$\lim_{b\downarrow 0,\ B\uparrow\infty} \lim_{n\to\infty} \frac{n}{h_{x_n}(1)} \sum_{bc_n<y\le Bc_n} P(S_n = x - y, T_x > n)\overline{F}(y) = C, \tag{48}$$

uniformly for $x_n \in [D^{-1}, D]$, since this would contradict (3) if C differed from 1. But in fact (48) follows immediately from (26) and the identity (31) in Proposition 15. When $\alpha\rho = 1$, it is immediate that

$$n\sum_{y>\delta_n c_n} P(S_n = x - y, T_x > n)\overline{F}(y) \le n\overline{F}(\delta_n c_n) \to 0,$$

and

$$\begin{aligned} & n\sum_{0\le y\le\delta_n c_n} P(S_n = x - y, T_x > n)\overline{F}(y) \\ \backsim\ & \frac{nP(\tau^- > n)p(x_n)}{c_n} \sum_{0\le y\le\delta_n c_n} V(y)\overline{F}(y) \\ \backsim\ & \frac{np(x_n)\omega(n)}{P(\tau > n)nc_n} \backsim \frac{p(x_n)P(\tau = n)}{P(\tau > n)} \backsim \rho p(x_n), \end{aligned}$$

and the result follows from the identity (33) in Proposition 15, since again there would be a contradiction if $\rho k_9 \ne 1$. ■

5.3 Proof when $x/c_n \to \infty$

Proof. This time we write $P(T_x = n+1) = \sum_1^4 P^{(i)}$, where $P^{(i)} = P\{A^{(i)} \cap (T_x = n+1)\}$ and

$$\begin{aligned} A^{(1)} &= (S_n \le \delta x),\ A^{(2)} = (\delta x < S_n \le x - Kc_n), \\ A^{(3)} &= (x - Kc_n < S_n \le x - \gamma c_n),\ \text{and } A^{(4)} = (S_n > x - \gamma c_n). \end{aligned}$$

We note first that for $\delta \in (0,1)$,

$$\begin{aligned} \limsup_{n\to\infty} \frac{P^{(1)}}{\overline{F}(x)} &\le \limsup_{n\to\infty} \frac{P(S_n \le \delta x)\overline{F}((1-\delta)x)}{\overline{F}(x)} = (1-\delta)^{-\alpha}, \\ \liminf_{n\to\infty} \frac{P^{(1)}}{\overline{F}(x)} &\ge \liminf_{n\to\infty} \frac{P(\max_{r\le n} S_r \le x, |S_n| \le \delta x)\overline{F}((1+\delta)x)}{\overline{F}(x)} \\ &= (1+\delta)^{-\alpha}. \end{aligned} \tag{49}$$

Next, using (27),

$$\begin{aligned} \limsup_{n\to\infty} \frac{P^{(2)}}{\overline{F}(x)} &\le \limsup_{n\to\infty} \frac{P(S_n > \delta x)\overline{F}(Kc_n)}{\overline{F}(x)} \\ &\le \limsup_{n\to\infty} \frac{n\overline{F}(\delta x)\overline{F}(c_n)}{\overline{F}(x)K^\alpha} = \frac{C}{(\delta K)^\alpha}. \end{aligned} \tag{50}$$

To deal with the next term, we use (27) again to see that for any fixed K, $P(x - Kc_n < S_n \le x)$ is uniformly $o(n\overline{F}(x))$. Since $P^{(3)} \le \overline{F}(\gamma c_n)P(x - Kc_n < S_n \le x)$, we deduce that

$$\frac{P^{(3)}}{\overline{F}(x)} \to 0 \text{ uniformly for each fixed } \gamma \text{ and } K. \tag{51}$$

As we are in the lattice case, (11) tells us that $f(x) := P(S_1 = x) \backsim \alpha x^{-1}\overline{F}(x)$, and combining this with (30) gives $g(n,x) = P(S_n = x, \tau^- > n) \backsim \alpha n P(\tau^- > n) x^{-1}\overline{F}(x)$, so we can assume that

$$\sup_{n>0,\ x>Kc_n} \frac{xg(n,x)P(\tau > n)}{\overline{F}(x)} < \infty. \tag{52}$$

Then for large enough n and $x > Kc_n$

$$\begin{aligned} P^{(4)} &= \sum_{0}^{n} \sum_{y=0}^{\lfloor \gamma c_n \rfloor} \sum_{z=0}^{\lfloor \gamma c_n \rfloor - y} g(n-m, x-y) g^-(m,z) \overline{F}(y+z) \\ &\le \frac{C\overline{F}(x)}{xP(\tau > n)} \sum_{0}^{n} \sum_{y=0}^{\lfloor \gamma c_n \rfloor} \sum_{z=0}^{\lfloor \gamma c_n \rfloor - y} g^-(m,z) \overline{F}(y+z) \\ &\le \frac{C\overline{F}(x)}{xP(\tau > n)} \sum_{y=0}^{\lfloor \gamma c_n \rfloor} \sum_{z=0}^{\lfloor \gamma c_n \rfloor - y} v(z) \overline{F}(y+z) \\ &= \frac{C\overline{F}(x)}{xP(\tau > n)} \sum_{z=0}^{\lfloor \gamma c_n \rfloor} \sum_{y=0}^{\lfloor \gamma c_n \rfloor - z} v(z) \overline{F}(y+z) \\ &= \frac{C\overline{F}(x)}{xP(\tau > n)} \sum_{z=0}^{\lfloor \gamma c_n \rfloor} \sum_{w=z}^{\lfloor \gamma c_n \rfloor} v(z) \overline{F}(w). \end{aligned}$$

Now a summation by parts and the fact that $V\overline{F} \in RV(-\alpha\rho)$ shows that as $y \to \infty$

$$\sum_{z=0}^{y} \sum_{w=z}^{y} v(z)\overline{F}(w) \backsim \sum_{z=0}^{y} V(z)\overline{F}(z) \backsim yV(y)\overline{F}(y)/(1-\alpha\rho).$$

So for all large enough n we have the bound

$$\begin{aligned} P^{(4)} &\le \frac{C\overline{F}(x)\gamma^{1-\alpha\rho} c_n V(c_n)}{nxP(\tau > n)} \\ &\le \frac{C\overline{F}(x)\gamma^{1-\alpha\rho} c_n}{nxP(\tau > n)P(\tau^- > n)} \backsim \frac{C\gamma^{1-\alpha\rho} c_n}{x} \cdot \overline{F}(x) \end{aligned} \tag{53}$$

The result follows from (49)-(53) and appropriate choice of δ, K, and γ. ■

Remark 16 *The assumption (11) is not strictly necessary for (52) to hold, and this is the only point where we use this assumption. In fact, if the following slightly weaker version of (52),*

$$\sup_{n>0,\ x>Kc_n} \frac{c_n g(n,x) P(\tau > n)}{\overline{F}(x)} < \infty, \tag{54}$$

were to hold, then (53) would hold with c_n replacing x in the denominator, and the proof would still be valid, by choosing γ small.

6 The non-lattice case

We indicate here the main differences between the proof in the lattice and non-lattice cases. First, we have

$$\begin{aligned} G(n,dy) &:= \sum_{r=0}^{\infty} P(T_r=n, H_r\in dy) = P(S_n\in dy, \tau^->n), \\ G^-(n,dy) &:= \sum_{r=0}^{\infty} P(T_r^-=n, H_r^-\in dy) = P(-S_n\in dy, \tau>n), \end{aligned}$$

and the following analogue of Lemma 10 is given in Theorems 3 and 4 of [21].

Lemma 17 *For any $\Delta_0>0$, uniformly in $x\geq 0$ and $0<\Delta\leq\Delta_0$,*

$$\frac{c_nG(n,[x,x+\Delta])}{P(\tau^->n)} = \Delta p(x/c_n)+o(1) \text{ and } \frac{c_nG^-(n,[x,x+\Delta])}{P(\tau>n)} = \Delta\tilde{p}(x/c_n)+o(1) \text{ as } n\to\infty. \tag{55}$$

Also, uniformly as $x/c_n\to 0$,

$$G(n,[x,x+\Delta]) \backsim \frac{f(0)\int_x^{x+\Delta}U(w)dw}{nc_n} \text{ and } G^-(n,[x,x+\Delta]) \backsim \frac{f(0)\int_x^{x+\Delta}V(w)dw}{nc_n}. \tag{56}$$

Remark 18 *Again, only the results for G are given in [21], but it is easy to get the reults for G^-. Actually the result in [21] has $U(w-)$ rather than $U(w)$ in (56), but clearly the two integrals coincide. Finally the uniformity in Δ is not mentioned in [21], but a perusal of the proof shows that this is true, essentially because it holds in Stone's local limit theorem. See e.g. Theorem 8.4.2 in [6].*

In writing down the analogues of (16) and (17) care is required with the the limits of integration, since the distribution of S_n and the renewal measures are not necessarily diffuse. These analogues are

$$P(T_x=n+1) = \int_{[0,\infty)} P(S_n\in x-dy, T_x>n)\overline{F}(y), \tag{57}$$

and for $w\geq 0$

$$P(S_n\in x-dw, T_x>n) = \sum_{r=0}^{n}\int_{[0,x)\cap[0,w]} G(r,x-dz)G^-(n-r,dw-z). \tag{58}$$

The key result, the analogue of Proposition 12, is

Proposition 19 *Fix $\Delta>0$. Then (i) uniformly as $x_n\vee y_n\to 0$,*

$$P(S_n\in(x-y-\Delta,x-y], T_x>n) \backsim \frac{U(x)f(0)\int_y^{y+\Delta}V(w)dw}{nc_n}. \tag{59}$$

(ii) For any $D>1$, uniformly for $y_n \in [D^{-1}, D]$,

$$P(S_n \in (x-y-\Delta, x-y], T_x > n) \backsim \frac{U(x)P(\tau > n)\Delta \tilde{p}(y_n)}{c_n} \text{ as } n \to \infty \text{ and } x_n \to 0, \tag{60}$$

and uniformly for $x_n \in [D^{-1}, D]$,

$$P(S_n \in (x-y-\Delta, x-y], T_x > n) \backsim \frac{V(y)P(\tau^- > n)\Delta p(x_n)}{c_n} \text{ as } n \to \infty \text{ and } y_n \to 0. \tag{61}$$

(iii) For any $D>1$, uniformly for $x_n \in [D^{-1}, D]$ and $y_n \in [D^{-1}, D]$,

$$P(S_n \in (x-y-\Delta, x-y], T_x > n) \backsim \frac{\Delta q_{x_n}(y_n)}{c_n} \text{ as } n \to \infty. \tag{62}$$

Once we have these results, we deduce Theorem 2 by applying them to a modified version of (57), viz

$$\begin{aligned} \sum_0^\infty P(S_n &\in (x-(n+1)\Delta, x-n\Delta], T_x > n)\overline{F}(n\Delta) \geq P(T_x = n+1) \\ &\geq \sum_0^\infty P(S_n \in (x-(n+1)\Delta, x-n\Delta], T_x > n)\overline{F}((n+1)\Delta), \end{aligned}$$

and letting $\Delta \to 0$. So the key step is establishing Proposition 19, and we illustrate how this can be done by proving (59).

Proof. We want to apply Lemma 17 to (58), but technically the problem is that we can't do this directly, as we did in the lattice case. The first step is to get an integrated form of (58), and it is useful to separate off the term $r=0$, so that for $x, y \geq 0$,

$$P(S_n \in (x-y-\Delta, x-y], T_x > n) = G^-(n, [\{y-x\}^+, y+\Delta-x))\mathbf{1}_{\{x \leq y+\Delta\}} + \tilde{P}, \tag{63}$$

where

$$\begin{aligned} \tilde{P} &= \sum_{r=1}^n \int_{y \leq w < y+\Delta} \int_{z \in [0,x) \cap [0,w]} G(r, x-dz)G^-(n-r, dw-z), \\ &= \sum_{r=1}^n \int_{0 \leq z < x \wedge (y+\Delta)} \int_{y \vee z \leq w < (y+\Delta)} G(r, x-dz)G^-(n-r, dw-z) \\ &= \sum_{r=1}^n \int_{0 \leq z < x \wedge (y+\Delta)} G(r, x-dz)G^-(n-r, [(y-z)^+, y+\Delta-z)), \end{aligned} \tag{64}$$

Using a similar notation as in the proof of Proposition 12 we split $\tilde{P}$ into three terms, and note first from (56) and (64) that

$$\begin{aligned} \tilde{P}_1 &\backsim f(0) \sum_{r=1}^{\lfloor n\delta \rfloor} \frac{1}{d(n-r)} \int_{0 \leq z < x \wedge (y+\Delta)} G(r, x-dz) \int_{(y-z)^+}^{y+\Delta-z} V(u)du \\ &\leq \frac{f(0)}{d(n(1-\delta))} \int_{0 \leq z < x \wedge (y+\Delta)} U(x-dz) \int_{(y-z)^+}^{y+\Delta-z} V(u)du. \end{aligned}$$

An asymptotic lower bound is given by

$$\frac{f(0)}{d(n)}\sum_{r=1}^{\lfloor n\delta\rfloor}\int_{0\le z<x\wedge(y+\Delta)}G(r,x-dz)\int_{(y-z)^+}^{y+\Delta-z}V(u)du,$$

and it is easy to see that

$$\sum_{r>n\delta}\int_{0\le z<x\wedge(y+\Delta)}G(r,x-dz)\int_{(y-z)^+}^{y+\Delta-z}V(u)du=o(U(x)V_\Delta(y)),$$

where we have put $V_\Delta(y):=\int_y^{y+\Delta}V(u)du$. Noting that $U(x-dz)=\sum_1^\infty G(r,x-dz)$ for $0\le z<x$, this leads to a similar uniform asymptotic lower bound, and hence that

$$\lim_{n,\delta}\frac{d(n)P_1}{f(0)\int_{0\le z<x\wedge(y+\Delta)}U(x-dz)\int_{(y-z)^+}^{y+\Delta-z}V(u)du}=1. \tag{65}$$

Dealing with $\tilde{P}_3$ is more complicated. First we write

$$\begin{aligned}\tilde{P}_3 &= \sum_{r=0}^{\lfloor n\delta\rfloor}\int_{0\le z<x\wedge(y+\Delta)}G(n-r,x-dz)G^-(r,[(y-z)^+,y+\Delta-z))\\ &= \sum_{r=0}^{\lfloor n\delta\rfloor}\int_{x\wedge(y+\Delta)-x<w\le x}G(n-r,dw)G^-(r,[(y-x+w)^+,y+\Delta-x+w)).\end{aligned}$$

We approximate this below and above by breaking the range of integration into subintervals of length $\varepsilon\ll\Delta$, then use the estimate $G(n-r,[k\varepsilon,(k+1)\varepsilon))\backsim f(0)\int_{k\varepsilon}^{(k+1)\varepsilon}U(v)dv/d(n-r)$, and finally let $\varepsilon\to0$ to conclude that

$$\lim_{n,\delta}\frac{d(n)P_3}{f(0)\int_{0\le z<x\wedge(y+\Delta)}U(x-z)dz\int_{y\vee z\le w<y+\Delta}V(du-z)}=1. \tag{66}$$

(Note that the term corresponding to $r=n$ in (64) is included here.)

Also, for any fixed $\delta\in(0,1/2)$ we can use (56) twice to see that

$$\begin{aligned}\tilde{P}_2 &\backsim f(0)\sum_{\lfloor n\delta\rfloor<r<\lfloor n(1-\delta)\rfloor}\frac{1}{d(n-r)}\int_{0\le z<x\wedge(y+\Delta)}G(r,x-dz)\int_{(y-z)^+}^{y+\Delta-z}V(u)du\\ &\le \frac{CV_\Delta(y)}{d(n)}\sum_{\lfloor n\delta\rfloor<r<\lfloor n(1-\delta)\rfloor}\int_{0\le z<x}G(r,x-dz)\\ &\le \frac{C}{d(n)}\sum_{\lfloor n\delta\rfloor<r<\lfloor n(1-\delta)\rfloor}\sum_{m=0}^{\lfloor x\rfloor}G(r,[m,m+1)\\ &\backsim \frac{CV_\Delta(y)}{d(n)}\sum_{\lfloor n\delta\rfloor<r<\lfloor n(1-\delta)\rfloor}\sum_{m=0}^{\lfloor x\rfloor}\frac{\int_m^{m+1}U(v)dv}{d(r)}\\ &\le \frac{Cx}{c_n}\cdot\frac{V_\Delta(y)U(x+1)}{d(n)}=o(\frac{V_\Delta(y)U(x)}{d(n)}).\end{aligned} \tag{67}$$

After reading off the asymptotic behaviour of the first term in (64) from (56), the proof is now completed by using (65), (66), (67), and the following result. ■

Lemma 20 *For $x, y \geq 0$ and $\Delta > 0$ the following identity holds*

$$\begin{aligned}\int_{0\leq z<x\wedge(y+\Delta)}\int_{y\vee z\leq w<y+\Delta} U(x-z)dzV(dw-z) \quad &+ \quad U(x-dz)V(w-z)dw \\ +\mathbf{1}_{\{x\leq y+\Delta\}}\int_{(y-x)^+}^{y+\Delta-x} V(w)dw \quad &= \quad U(x)V_\Delta(y). \qquad (68)\end{aligned}$$

Proof. Assume first that $y \geq x$, so that $x \wedge (y+\Delta) = x$, and the first integral reduces to

$$\begin{aligned}&\int_{0\leq z<x}\int_{y\leq w<y+\Delta} U(x-z)dzV(dw-z) + U(x-dz)V(w-z)dw \\ = \quad &\int_{0\leq z<x} U(x-z)[V((y-z+\Delta)-) - V((y-z)-)]dz + U(x-dz)V_\Delta(y-z) \\ = \quad &\int_{0\leq z<x} -\frac{d}{dz}[U(x-z)V_\Delta(y-z)]du = U(x)V_\Delta(y) - U(0)V_\Delta(y-x).\end{aligned}$$

This verifies (68), since $U(0) = 1$ and the second term on the LHS of (68) is $V_\Delta(y-x)$ when $y \geq x$. If $y < x$ we split the first integral into two parts and repeat the above calculation to see that

$$\begin{aligned}&\int_{0\leq z<y}\int_{y\leq w<y+\Delta} U(x-z)dzV(dw-z) + U(x-dz)V(w-z)dw \\ = \quad &U(x)V_\Delta(y) - U(x-y)V_\Delta(0). \qquad (69)\end{aligned}$$

The second part is, writing $\overline{V}(z) = \int_0^z V(w)dw$,

$$\begin{aligned}&\int_{y\leq z<x\wedge(y+\Delta)}\int_{z\leq w<y+\Delta} U(x-z)dzV(dw-z) + U(x-dz)V(w-z)dw \\ = \quad &\int_{y\leq z<x\wedge(y+\Delta)} U(x-z)V((y+\Delta-z)-)dz + U(x-dz)\overline{V}(y+\Delta-z) \\ = \quad &\int_{y\leq z<x\wedge(y+\Delta)} -\frac{d}{dz}[U(x-z)\overline{V}(y+\Delta-z)] \\ = \quad &U(x-y)V_\Delta(0) - U(x-(x\wedge(y+\Delta))\overline{V}(y+\Delta-(x\wedge(y+\Delta)) \\ = \quad &U(x-y)V_\Delta(0) - \overline{V}(y+\Delta-x)\mathbf{1}_{\{y+\Delta>x\}}. \qquad (70)\end{aligned}$$

Since the second term in (68) reduces to $\overline{V}(y+\Delta-x)\mathbf{1}_{\{y+\Delta>x\}}$ when $y < x$, the proof in this case follows from (69) and (70). ■

Remark 21 *The recent paper [1] contains some functional limit theorems for conditional random walks in the domain of attraction of a one-sided stable law.*

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